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LETTER TO THE EDITOR

Phase transitions in driven diffusive systems with random rates

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Abstract. We study a one-dimensional driven lattice gas model in which quenched random jump rates are associated with the particles. Under suitable conditions on the distribution of jump rates the model displays a phase transition from a high-density 'laminar' phase with product measure to a low-density 'jammed' phase in which the interparticle spacings have no stationary distribution. Using a waiting time representation the phase transition is shown to be equivalent to a pinning transition of directed polymers with columnar defects. The phenomenon is argued to have a natural realization in traffic flow.

A set of equivalent problems including driven diffusion [1], interface motion [2] and directed paths (or polymers) in random media [3] has recently reached a paradigmatic status in non-equilibrium statistical physics [4]. In this letter we show how these systems react to the presence of a particular type of quenched disorder. Our starting point is a class of exactly-solved models introduced by Benjamini, Ferrari and Landim (BFL) [5, 6]. We simplify the BFL model to allow for explicit analytic computations which are supplemented with computer simulations, and interpret the results in the diverse contexts mentioned above.

Let us first provide an intuitive picture of the basic phenomenon. Imagine traffic on a single-lane highway with no possibility of passing. Different cars have different preferred speeds—compare e.g. a Ferrari to a 2CV. In the absence of passing it is clear that the overall speed will be set by the slowest cars. However, at high traffic density all cars have to move more slowly than their preferred speeds, and the differences between the cars are irrelevant. In contrast, at low density, jams will form behind the slow cars. Our central result implies that, under suitable conditions on the distribution of preferred speeds, a *sharp phase transition* separates the high-density 'laminar' phase from the low-density 'jammed' phase. Quantitatively, the two phases are characterized by the fluctuations in the distances between cars, which are bounded in the laminar phase but diverge in the jammed phase; more precisely, the spacings between cars cease to have a (normalizable) stationary distribution. At the transition point the distribution has a power-law tail. As will be detailed below, the jammed phase has been studied previously in the contexts of interface motion with random growth rates [7] and of directed polymers with columnar disorder [8, 9].

The model we wish to study is a modification of the basic one-dimensional driven lattice gas, known in the mathematical literature as the (totally) asymmetric simple exclusion process (ASEP) [10, 11]; related models have been extensively used for the simulation of traffic flow [12]. Each site x of a one-dimensional lattice with periodic boundary conditions (a ring) is either empty or occupied by one particle ('car'). Particles move to the right to vacant nearest-neighbour sites, according to exponentially distributed waiting times; this

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corresponds to a random sequential updating procedure. Let x_i denote the position of the ith particle and p_i its jump rate (the labelling is preserved by the dynamics). In the standard ASEP $p_i \equiv p$ for all i.

In the general case, where the p_i depend on i, one can only hope for the interparticle distances to become stationary, rather than, for example, the density at a fixed site. We, therefore, introduce the variables $u_i = x_{i+1} - x_i - 1$ to denote the size of the gap (the number of vacant sites) in front of the ith particle. The dynamics of the u_i defines a 'zero-range process' [10, 11]: if we regard u_i as a particle occupation number on an auxiliary lattice, then a u-particle at site i jumps to site $i-1$ at rate p_i independent of the state at site $i-1$; in this sense, the u-particles have zero interaction range.

It is well known [10, 11, 13] that zero-range processes have product stationary measures even when the transition rates are not translationally invariant. In the present case this implies that the product of the exponential distributions

$$\text{Prob}[u_i = k] = (1-\alpha_i)\alpha_i^k \tag{1}$$

satisfies the stationarity condition, provided

$$p_i \alpha_i = \text{constant} \equiv v \tag{2}$$

for all i, where v is the (common) average velocity of the particles.

Of course, this solution is viable only if (1) is normalizable, i.e. $\alpha_i < 1$ for all i, or

$$v < c \equiv \min_i \{p_i\}. \tag{3}$$

The actual value of v is determined by the average particle density ρ, which is conserved by the dynamics. The mechanism which leads to the possibility of a phase transition in the model can now be described as follows. For a given density ρ, we use the product measure (1) to compute the velocity $v(\rho)$. If $v < c$ the solution is valid and the system is in the laminar state, with bounded fluctuations and no correlations. If, on the other hand, it is found that $v(\rho^*) = c$ for some $\rho^* > 0$, then the product measure solution breaks down and the system enters a jammed state at low densities $\rho < \rho^*$.

The existence of a phase transition at a critical density ρ^* was established by BFL for a more general case, in which particles jump backwards at rate q_i. The stationarity condition (2) then becomes $p_i\alpha_i - q_i\alpha_{i-1} = v$, which leads to a complicated expression for the α_i [5, 6]. The fully asymmetric model is, therefore, better suited for explicit computations, at the expense of more restrictive conditions on the distribution of jump rates [5].

For the purpose of illustration, consider first the case of a single slow 'defect' particle, $p_1 = c < 1$ and $p_i = 1$ for $i \neq 1$ [14]. Then the average velocity in the laminar state is determined by the majority of fast particles, and is given by the expression $v(\rho) = 1 - \rho$ of the pure system. The laminar state breaks down at $\rho^* = 1 - c$. While the mathematical characterization of the jammed state is non-trivial, it is intuitively obvious what will happen: for $\rho < \rho^*$ the fast particles pile up behind the defect, forming a jam of density ρ^*. In a finite system of L sites a steady state is reached which has a gap of size $L(1 - \rho/\rho^*)$ in front of the slow particle, while in the infinite system the gap grows with time as $(\rho^* - \rho)t$ and the density in front of the gap remains at ρ.

The focus of this paper is on the case where p_i are independent random variables drawn from a common distribution $f(p)$ with support on the interval $[c, 1]$, $c > 0$. Here the velocity $v(\rho)$ is determined implicitly through the relation

$$\frac{1-\rho}{\rho} = \overline{\langle u_i \rangle} = v \int_c^1 \frac{\mathrm{d}p\, f(p)}{p - v} \tag{4}$$

which follows by computing the average gap size (for a fixed set of rates) $\langle u_i \rangle = \alpha_i/(1-\alpha_i)$ using (1), and then performing the disorder average denoted by the overbar. Similarly the stationary variance of the gap size can be expressed as

$$\Delta^2 \equiv \overline{\langle u_i^2 \rangle - \langle u_i \rangle^2} = -v\rho^2 \left(\frac{\mathrm{d}v}{\mathrm{d}\rho} \right)^{-1}. \tag{5}$$

The occurrence of a phase transition at some finite density ρ^* is related to the existence of the integral on the right-hand side of (4) at $v = c$, if the integral is finite, its value equals $(1 - \rho^*)/\rho^*$ and a transition occurs. Thus, a transition is possible only if $f(p)$ vanishes sufficiently rapidly for $p \to c$. Introducing an exponent n such that

$$f(p) \sim (p - c)^n \qquad p \to c \tag{6}$$

the various behaviours can be classified as follows. (i) For $n \leqslant 0$ there is no transition at finite density; however, in general, there is a singularity at $\rho = 0$, e.g. for $n = 0$ the gap size fluctuations diverge as $\Delta^2 \sim \exp(1/\rho)$ for $\rho \to 0$. (ii) For $0 < n \leqslant 1$ the transition is of *second order* in the sense that the first derivative $\mathrm{d}v/\mathrm{d}\rho$ vanishes as $\rho \to \rho^*$ from above, and correspondingly $\Delta^2(\rho)$ diverges (cf equation (5)), as $\Delta^2 \sim (\rho - \rho^*)^{-(1-n)/n}$. An example of this case is shown in figure 1. (iii) For $n > 1$ the transition is of *first order*, i.e. $\mathrm{d}v/\mathrm{d}\rho$ and Δ^2 have finite limits as $\rho \to \rho^*$ from above. (iv) For any $n > 0$ the gap distribution at the transition has a power-law tail; performing the disorder average of the stationary distribution (1) one finds that, for large k,

$$\overline{\mathrm{Prob}[u_i = k]}|_{\rho=\rho^*} \sim k^{-(n+2)} \tag{7}$$

and consequently moments of order higher than $n + 1$ diverge for $\rho \to \rho^*$.

Since the product measure solution breaks down at $\rho = \rho^*$, it does not give any information about the nature of the jammed phase; nevertheless a plausible scenario can be gathered from heuristic reasoning and computer simulations. First, note that due to the no-passing constraint the overall velocity can never exceed c, and hence $v(\rho) \equiv c$ for all $\rho < \rho^*$. This is confirmed by the numerical data for $v(\rho)$ shown in the inset of figure 1. The stationary state in a finite system is dominated by the slowest particle and thus has the same structure as in the single defect case, with a single large gap trailed by a jam of density ρ^*. In the simulations this is easily checked by measuring the stationary value of Δ^2, which is given by $\Delta^2 \approx (L/\rho)(1 - \rho/\rho^*)^2$ for a single gap configuration (see the main part of figure 1).

The *approach* to this stationary state, which reflects the behaviour of the infinite system, is more complicated. Simulations show the appearance of many small gaps in the system, which subsequently undergo a *coarsening* process characterized by a typical gap spacing $\xi(t)$ (see inset of figure 2). Under the assumption that the gap distribution can be regarded as a superposition of small gaps (corresponding to the 'laminar' regions of density ρ^*) and large gaps (regions of density zero), it follows that $\Delta^2(t) \sim \xi(t)$. The simulation data shown in figure 2 indicate a power-law coarsening, $\xi(t) \sim t^{1/z}$, with a *non-universal* dynamic exponent z which increases with increasing n.

An approximate picture for the long-time coarsening dynamics can be obtained from an extremal statistics estimate. The particles which cause the jams at time t are obviously the slowest particles in regions of size $\xi(t)$. Using the behaviour of the distribution (6) close to the minimal velocity, it can be shown [15] that the velocity differences of these 'extremal' particles are of the order $\Delta v \sim 1/\xi^{1/(n+1)}$. The time required for two jams to merge is then of the order $\xi/\Delta v$, which implies a dynamic exponent $z = (n + 2)/(n + 1)$. While this expression gives the correct trend, it exceeds the numerical values for $n < 1$ and

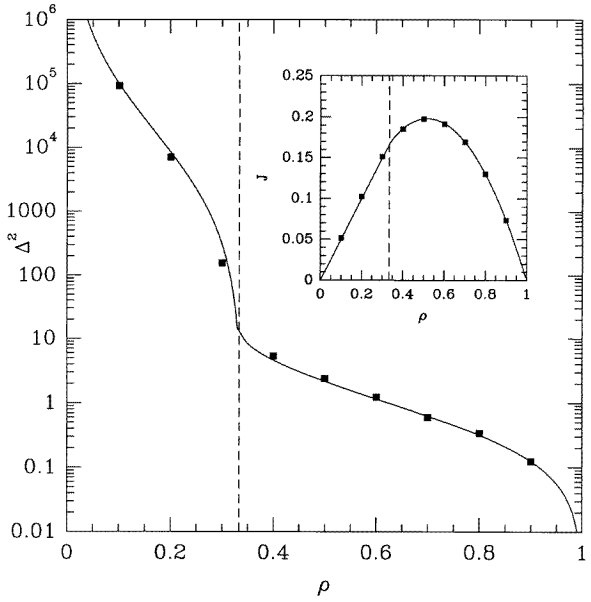

Figure 1. Numerical (squares) and analytic (full curves) results for the variance of the interparticle spacing (main figure) and the particle current (inset) for the random rate model with distribution $f(p) = [(n+1)/(1-c)^{n+1}](p-c)^n$ with $n = 1$ and $c = \frac{1}{2}$. For this case a second-order jamming transition is predicted to occur at density $\rho^* = n(1-c)/(n+c) = \frac{1}{3}$ (vertical dashed lines). The variance Δ^2 has a logarithmic divergence at $\rho = \rho^*$; in the jammed phase the full curve represents the expression expected for a single gap configuration. The simulations were carried out with 2000 particles simulated over 10^6 attempted jumps per particle.

underestimates them for $n > 1$ (see figure 2). Possibly this is related to the fact that the 'laminar' phase itself has large gap fluctuations at $\rho = \rho^*$ (compare to (7)).

The behaviour of traffic models is usually characterized by the *fundamental diagram*, giving the current or flow $J(\rho) = \rho v(\rho)$ as a function of density [12]. In this representation the jammed phase corresponds to a linear segment of slope $\mathrm{d}J/\mathrm{d}\rho = c$ for $\rho < \rho^*$. The critical density ρ^* is generally different from the density $\rho_{\max}$ corresponding to maximal flow, $\mathrm{d}J/\mathrm{d}\rho|_{\rho_{\max}} = 0$. For the second-order case it is easy to see that $\rho^* < \rho_{\max}$ always (see inset of figure 1), while for the first-order case $\rho^* < \rho_{\max}$ or $\rho^* = \rho_{\max}$ depending on the parameters.

Next we outline how our model is related to the other problems mentioned in the introduction. The mapping between exclusion models and moving interfaces is by now standard [11, 16]. With each particle we associate a 'height' variable $h_i(t)$ which counts the number of jumps the particle has performed, starting from $h_i(0) = x_i(0) - i$. This defines an interface $j = h_i$ on the square lattice of points (i, j), which is a staircase with step heights $u_i = h_{i+1} - h_i \geqslant 0$ and average slope $(1-\rho)/\rho$ (figure 3; the interface representation is also used in figure 2). Moving the ith particle increases h_i by one. Thus we are dealing with an *inhomogeneous deposition* problem [17], with deposition rate p_i in the ith lattice column, for which the stationary interface shape can be determined *exactly* for any choice of rates which do not violate condition (3). The phase transitions discussed above correspond to *faceting* transitions in which the interface develops facets of infinite

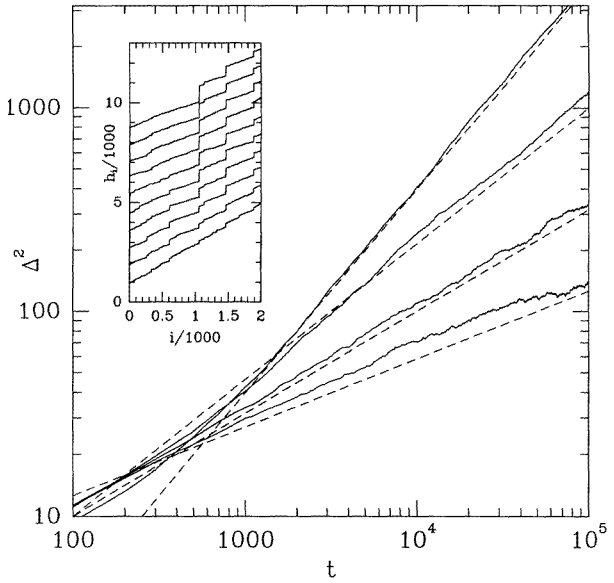

Figure 2. Transient behaviour of the variance Δ^2 for a large system (10^5 particles) in the jammed phase at density $\rho = \frac{1}{3}$. The distribution $f(p)$ was chosen as in figure 1, however with $c = \frac{1}{3}$. The full curves show numerical results for (from top to bottom) $n = 2, n = 1, n = \frac{1}{2}$ and $n = \frac{1}{3}$, respectively. The dashed lines indicate power laws with exponents $1/z = 1 (n = 2)$, $\frac{2}{3} (n = 1)$, $\frac{1}{2}$ $(n = \frac{1}{2})$ and $\frac{1}{3} (n = \frac{1}{3})$. Inset: time evolution of 'height' configurations of 2000 particles for $n = 2$; note the appearance of large gaps and their subsequent coarsening.

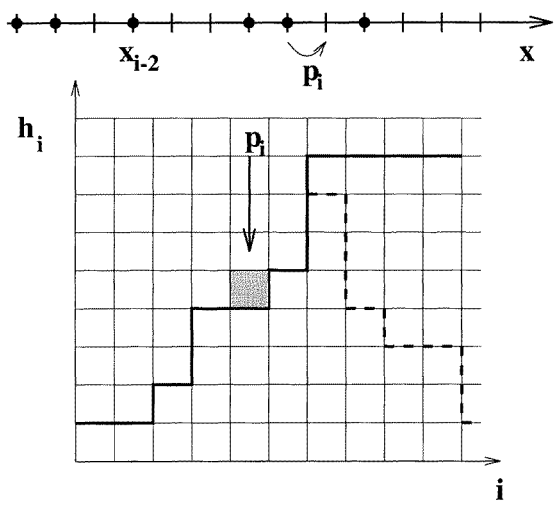

Figure 3. Illustration of the mapping to interface motion and optimal paths. The bold line shows the interface configuration corresponding to the particle configuration shown above. The bold dashed line shows an allowed directed path.

slope (the gaps) beyond a critical average inclination $u^* = (1 - \rho^*)/\rho^*$. Inhomogeneous deposition with quenched random rates was previously investigated in [7]; while some of

the models considered there can also be mapped onto lattice gases, the resulting exclusion models have random rates associated with *sites* rather than particles, which leads to a much more difficult problem [5] (see below; in the context of traffic models this kind of disorder has been considered in [18]).

The interface model can be transformed into an optimal path problem using the waiting time formulation of the growth process [19]. In this approach one introduces the time $t(i, j)$ at which the interface first reaches a point (i, j) of the square lattice. It is then easily seen that the quantity $E(i, j) \equiv -t(i, j)$ is the solution of the following optimization problem. Consider all paths which are directed along the positive j- and negative i-directions, 'transverse' to the interface (figure 3). Associate with each point of the lattice a random energy $\epsilon(i, j)$, drawn from an exponential distribution with mean $\langle \epsilon(i, j) \rangle = -1/p_i$ (these are simply the negative random waiting times of the stochastic model). Define the energy of a path to be the sum of the site energies encountered along the path; then $E(i, j)$ is the energy of the *optimal* (lowest energy) path that ends at (i, j).

Since the average site energies $\langle \epsilon(i, j) \rangle$ depend on the column number i, the random disorder in the optimal path problem has a *columnar* component [8, 9]; slow particles (small p_i) correspond to particularly attractive *columnar defects*. The jamming transition can, therefore, be reinterpreted as a *pinning* transition, in which the columnar defects force the optimal path to align with the j-axis. The density in the lattice gas model can be shown to be related to the average orientation of the path enforced by the boundary conditions [20]. When the angle between the average orientation and the orientation of the defects becomes smaller than some critical value, it becomes favourable for the path to follow the defect over a finite fraction of its length†. The coarsening behaviour in the transient regime of the lattice gas model (figure 2) can be understood in terms of the 'evolutionary hopping' [8] of the optimal path—as the path becomes longer, more and more favourable defects become accessible, and the energy landscape (which corresponds, roughly, to the height configurations shown in figure 2 [19]) reflects the sudden transitions between different attractors.

Pinning transitions of optimal paths involving linear defects have attracted much attention recently [21–23], and only a few exact results are known even for single defects [19, 21]. In the present model the pinning by a single defect proceeds through a rather trivial first-order transition. We believe that the reason for the simplicity of our model, in the language of optimal paths, is due to the fact that a path can never return to a defect it has left (see figure 3); this seems to eliminate some of the subtleties associated with the competition between the attractive defect and the wandering induced by the bulk disorder.

Evidently an important question concerns the robustness of our results with respect to modifications of the model. For the single defect case Mallick [24] has recently shown that the transition persists if the slow defect particle is allowed to be overtaken, with a small rate, by the fast particles, although the solution becomes much more involved. In the context of traffic modelling [12] one would like to address the effects of parallel, rather than random sequential updating; at least for models with maximal velocity $v_{\max} = 1$ this is not expected to make a major difference, since these models can still be mapped onto optimal path problems with slightly different random energy distributions [20, 25]. Finally, the introduction of open boundary conditions might lead to an interesting interplay between boundary-induced [19, 26] and disorder-induced phase transitions.

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† This simple picture in fact applies only to the first-order transitions, see [20].

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Note added in proof. After this work was submitted for publication we became aware of two related papers. First, Ben-Naim *et al* [27] obtained the extremal statistics estimate of the dynamic exponent z within a deterministic traffic model with random velocities. Second, Nagatani [28] studied numerically the ASEP with parallel update and random rates associated with particles. In our notation he considered the case $n = 0$ (a uniform distribution) for which there should be no jammed state at finite density, but large gap fluctuations for $\rho \to 0$; this seems to be consistent with the numerical data [28].

References

[1] van Beijeren H, Kutner R and Spohn H 1985 *Phys. Rev. Lett.* **54** 2026
[2] Kardar M, Parisi G and Zhang Y C 1986 *Phys. Rev. Lett.* **56** 889
[3] Kardar M and Zhang Y-C 1987 *Phys. Rev. Lett.* **58** 2087
[4] Halpin-Healy T J and Zhang Y C 1995 *Phys. Rep.* **254** 215
[5] Benjamini I, Ferrari P A and Landim C 1996 *Stoch. Proc. Appl.* **61** 181
[6] After completion of this work we learned that some of the results of BFL and of the present paper have been independently obtained by M R Evans (preprint, cond-mat/9606036)
[7] Krug J 1995 *Phys. Rev. Lett.* **75** 1795; 1992 *Surface Disordering Growth, Roughening and Phase Transitions* ed R Jullien, J Kertész, P Meakin and D E Wolf (Commack, NY: Nova Science) p 177
[8] Krug J and Halpin-Healy T 1993 *J. Physique* I **3** 2179
[9] Arsenin I, Halpin-Healy T and Krug J 1994 *Phys. Rev.* E **49** R3561
[10] Spitzer F 1970 *Adv. Math.* **5** 246
[11] Spohn H 1991 *Large Scale Dynamics of Interacting Particles* (Berlin: Springer) and references therein
[12] Schreckenberg M, Schadschneider A, Nagel K and Ito N 1995 *Phys. Rev.* E **51** 2939
Nagel K 1996 *Phys. Rev.* E **53** 4655
[13] Andjel E D 1982 *Ann. Prob.* **10** 525
[14] Landim C 1996 *Ann. Prob.* at press
[15] Gumbel E J 1958 *Statistics of Extremes* (New York: Columbia University Press)
[16] Rost H 1981 *Z. Wahrsch verw Geb.* **58** 41
[17] Wolf D E and Tang L H 1990 *Phys. Rev. Lett.* **65** 1591
[18] Csahók Z and Vicsek T 1994 *J. Phys. A: Math. Gen.* **27** L591
[19] Krug J and Tang L H 1994 *Phys. Rev.* E **50** 104
[20] Krug J unpublished
[21] Kardar M 1985 *Phys. Rev. Lett.* **55** 2235
[22] Balents L and Kardar M 1994 *Phys. Rev.* B **49** 13 030
[23] Hwa T and Nattermann T 1995 *Phys. Rev.* B **51** 455
[24] Mallick K unpublished
see also Derrida B 1992 *Statphys 19* ed B-L Hao (Singapore: World Scientific) p 243
[25] Tang L-H 1992 *Growth Patterns in Physical Sciences and Biology* ed E Louis, L M Sander, P Meakin and J M Garcia-Ruiz (New York: Plenum)
[26] Krug J 1991 *Phys. Rev. Lett.* **67** 1882
[27] Ben-Naim E, Krapivsky P L and Redner S 1994 *Phys. Rev.* E **50** 822
[28] Nagatani T 1995 *Phys. Rev.* E **51** 922